

A Brezis–Browder theorem for SSDB spaces

by

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Abstract

In this paper, we show how the Brezis–Browder theorem for maximally monotone multifunctions with a linear graph on a reflexive Banach space, and a consequence of it due to Yao, can be generalized to SSDB spaces.

1 Introduction

This note is about generalizations of the following results of Brezis–Browder and Yao: *Let E be a nonzero reflexive Banach space with topological dual E^* and A be a norm-closed monotone linear subspace of $E \times E^*$. Then A is maximally monotone $\iff A^*$ is monotone $\iff A^*$ is maximally monotone.* In Theorem 4.4, we show that these results can be successfully generalized to SSDB space (see Section 2). The original results of Brezis–Browder and Yao appear in Corollary 4.5. Theorem 4.4 depends on a new transversality result, which appears in Theorem 3.3.

2 SSDB spaces

All vector spaces in the paper will be *real*. We will use the standard notation of convex analysis for “Fenchel conjugate” and “subdifferential” in a Banach space.

Definition 2.1. We will say that $(B, [\cdot, \cdot], q, \|\cdot\|, \iota)$ is a *symmetrically self-dual Banach space* (SSDB space) if B is a nonzero real vector space, $[\cdot, \cdot]: B \times B \rightarrow \mathbb{R}$ is a symmetric bilinear form, for all $b \in B$, $q(b) := \frac{1}{2}[b, b]$ (“ q ” stands for “quadratic”), $(B, \|\cdot\|)$ is a Banach space, and ι is a linear isometry from B onto B^* such that, for all $b, c \in B$, $\langle b, \iota(c) \rangle = [b, c]$. It is easily seen that if $(B, [\cdot, \cdot], q, \|\cdot\|, \iota)$ is a SSDB space then so also is $(B, -[\cdot, \cdot], -q, \|\cdot\|, -\iota)$. Clearly, for all $b, c \in B$,

$$\begin{aligned} |q(b) - q(c)| &= \frac{1}{2}|[b, b] - [c, c]| = \frac{1}{2}|[b - c, b + c]| = \frac{1}{2}|\langle b - c, \iota(b + c) \rangle| \\ &\leq \frac{1}{2}\|b - c\|\|\iota(b + c)\| = \frac{1}{2}\|b - c\|\|b + c\|. \end{aligned} \quad (1)$$

We refer the reader to Examples 2.2 and 2.3 below and [3] for various examples. There is also a construction that can be used for producing more pathological examples in [6, Remark 6.7, p. 20].

Let $(B, [\cdot, \cdot], q, \|\cdot\|, \iota)$ be an SSDB space and $A \subset B$. We say that A is *q -positive* if $A \neq \emptyset$ and $b, c \in A \implies q(b - c) \geq 0$. We say that A is *q -negative* if $A \neq \emptyset$ and $b, c \in A \implies q(b - c) \leq 0$. q -negativity is equivalent to $(-q)$ -positivity. We say that A is *maximally q -positive* if A is q -positive and A is not properly contained in any other q -positive set. In this case,

$$b \in B \implies \inf q(A - b) \leq 0. \quad (2)$$

Similarly, we say that A is *maximally q -negative* if A is q -negative and A is not properly contained in any other q -negative set. Maximal q -negativity is equivalent to maximal $(-q)$ -positivity.

A Brezis–Browder theorem for SSDB spaces

Let $g_0 := \frac{1}{2}\|\cdot\|^2$ on B . Then, for all $c \in B$, $g_0^*(\iota(c)) = \frac{1}{2}\|\iota(c)\|^2 = \frac{1}{2}\|c\|^2 = g_0(c)$. Consequently, for all $b, c \in B$,

$$\iota(c) \in \partial g_0(b) \iff g_0(b) + g_0^*(\iota(c)) = \langle b, \iota(c) \rangle \iff g_0(b) + g_0(c) = [b, c]. \quad (3)$$

We note from (1) with $c = 0$ that

$$g_0 + q \geq 0 \text{ on } B. \quad (4)$$

Examples 2.2.

(a) If B is a Hilbert space with inner product $(b, c) \mapsto \langle b, c \rangle$ and the Hilbert space norm then $(B, [\cdot, \cdot], q, \|\cdot\|, \iota)$ is a SSDB space with $[b, c] := \langle b, c \rangle$, $q = g_0$ and $\iota(c) := c$, and every nonempty subset of B is q -positive.

(b) If B is a Hilbert space with inner product $(b, c) \mapsto \langle b, c \rangle$ and the Hilbert space norm then $(B, [\cdot, \cdot], q, \|\cdot\|, \iota)$ is a SSDB space with $[b, c] := -\langle b, c \rangle$, $q = -g_0$ and $\iota(c) := -c$, and the q -positive sets are the singletons.

(c) If $B = \mathbb{R}^3$ under the Euclidean norm and

$$[(b_1, b_2, b_3), (c_1, c_2, c_3)] := b_1c_2 + b_2c_1 + b_3c_3,$$

then $(B, [\cdot, \cdot], q, \|\cdot\|, \iota)$ is a SSDB space, $q(b_1, b_2, b_3) = b_1b_2 + \frac{1}{2}b_3^2$ and $\iota(c_1, c_2, c_3) := (c_2, c_1, c_3)$. Here, if M is any nonempty monotone subset of $\mathbb{R} \times \mathbb{R}$ (in the obvious sense) then $M \times \mathbb{R}$ is a q -positive subset of B . The set $\mathbb{R}(1, -1, 2)$ is a q -positive subset of B which is not contained in a set $M \times \mathbb{R}$ for any monotone subset of $\mathbb{R} \times \mathbb{R}$. The helix $\{(\cos \theta, \sin \theta, \theta) : \theta \in \mathbb{R}\}$ is a q -positive subset of B , but if $0 < \lambda < 1$ then the helix $\{(\cos \theta, \sin \theta, \lambda \theta) : \theta \in \mathbb{R}\}$ is not.

Example 2.3. Let E be a nonzero reflexive Banach space with topological dual E^* . For all (x, x^*) and $(y, y^*) \in E \times E^*$, let $[(x, x^*), (y, y^*)] := \langle x, y^* \rangle + \langle y, x^* \rangle$. We norm $E \times E^*$ by $\|(x, x^*)\| := \sqrt{\|x\|^2 + \|x^*\|^2}$. Then $(E \times E^*, \|\cdot\|)^* = (E^* \times E, \|\cdot\|)$, under the duality $\langle (x, x^*), (y^*, y) \rangle := \langle x, y^* \rangle + \langle y, x^* \rangle$. It is clear from these definitions that $\langle (x, x^*), (y^*, y) \rangle = [(x, x^*), (y, y^*)]$. Thus $(E \times E^*, [\cdot, \cdot], q, \|\cdot\|, \iota)$ is a SSDB space with $q(x, x^*) = \frac{1}{2}[\langle x, x^* \rangle + \langle x, x^* \rangle] = \langle x, x^* \rangle$ and $\iota(y, y^*) = (y^*, y)$.

We now note that if $(x, x^*), (y, y^*) \in B$ then $\langle x - y, x^* - y^* \rangle = q(x - y, x^* - y^*) = q((x, x^*) - (y, y^*))$. Thus if $A \subset B$ then A is q -positive exactly when A is a nonempty monotone subset of B in the usual sense, and A is maximally q -positive exactly when A is a maximally monotone subset of B in the usual sense.

We define the reflection map $\rho_1: E \times E^* \rightarrow E \times E^*$ by $\rho_1(x, x^*) := (-x, x^*)$. Since $q \circ \rho_1 = -q$, a subset A of B is q -positive (resp. maximally q -positive) if, and only if, $\rho_1(A)$ is q -negative (resp. maximally q -negative). ρ_1 and its companion map ρ_2 are used in the discussion of an abstract Hammerstein theorem in [5, Section 30, pp. 123–125] and [7, Section 7, pp. 13–15].

A Brezis–Browder theorem for SSDB spaces

3 A transversality result

Let $B = (B, [\cdot, \cdot], q, \|\cdot\|, \iota)$ be a SSDB space. We write $\mathcal{PC}(B)$ for the set of all proper convex functions from B into $] -\infty, \infty]$ and $\text{dom } f$ for the set $\{x \in B: f(x) \in \mathbb{R}\}$. If $f \in \mathcal{PC}(B)$, we write f^\circledast for the Fenchel conjugate of f with respect to the pairing $[\cdot, \cdot]$, that is to say, for all $c \in B$, $f^\circledast(c) := \sup_B [\cdot, c] - f$. We note then that, for all $c \in B$,

$$f^\circledast(c) = \sup_B [\langle \cdot, \iota(c) \rangle - f] = f^*(\iota(c)). \quad (5)$$

We write $\mathcal{PCLSC}(B)$ for the set $\{f \in \mathcal{PC}(B): f \text{ is lower semicontinuous on } B\}$.

Lemma 3.1. *Let $(B, [\cdot, \cdot])$ be a SSDB space, $f \in \mathcal{PC}(B)$ and $c \in B$. We define $f_c \in \mathcal{PC}(B)$ by $f_c := f(\cdot + c) - [\cdot, c] - q(c)$. Then*

$$(f_c)^\circledast = (f^\circledast)_c \quad (6)$$

and, writing f_c^\circledast for the common value of these two functions,

$$\text{for all } b, d \in B, \quad f_c(b) + f_c^\circledast(d) - [b, d] = f(b + c) + f^\circledast(d + c) - [b + c, d + c]. \quad (7)$$

Proof. For all $b \in B$,

$$\begin{aligned} f_c^\circledast(b) &= \sup_{d \in B} [[d, b] + [d, c] + q(c) - f(d + c)] \\ &= \sup_{e \in B} [[e - c, b + c] + q(c) - f(e)] \\ &= \sup_{e \in B} [[e, b + c] - [c, b] - f(e)] - q(c) \\ &= f^\circledast(b + c) - [c, b] - q(c) = (f^\circledast)_c(b), \end{aligned}$$

which gives (6), and (7) follows since

$$\begin{aligned} f_c(b) + f_c^\circledast(d) - [b, d] &= f(b + c) - [b, c] - q(c) + f^\circledast(d + c) - [d, c] - q(c) - [b, d] \\ &= f(b + c) + f^\circledast(d + c) - [b, c] - [c, c] - [d, c] - [b, d] \\ &= f(b + c) + f^\circledast(d + c) - [b + c, d + c]. \end{aligned} \quad \square$$

Theorem 3.3 is the main result of this section, and uses the following consequence of Rockafellar's formula for the subdifferential of a sum (see [4, Theorem 3, p. 86]).

Lemma 3.2. *Let X be a nonzero normed space, $f: X \mapsto] -\infty, \infty]$ be convex on X , finite at a point of X , and $g: X \mapsto \mathbb{R}$ be convex and continuous. Then $\partial(f + g) = \partial f + \partial g$.*

Theorem 3.3. *Let $B = (B, [\cdot, \cdot], q, \|\cdot\|, \iota)$ be a SSDB space, $f \in \mathcal{PCLSC}(B)$ and, whenever $b, d \in B$,*

$$f(b) + f^\circledast(d) = [b, d] \implies q(b) + q(d) \leq [b, d]. \quad (8)$$

Let $\mathcal{N}_q(g_0) := \{b \in B: g_0(b) + q(b) = 0\}$. Then $\text{dom } f - \mathcal{N}_q(g_0) = B$.

A Brezis–Browder theorem for SSDB spaces

Proof. Let c be an arbitrary element of B . Now $f_c + g_0$ is convex, and $(f_c + g_0)(b) \rightarrow \infty$ as $\|b\| \rightarrow \infty$, i.e., $f_c + g_0$ is “coercive”. Since g_0 is continuous and convex, $f_c + g_0 \in \mathcal{PCLSC}(B)$, hence $f_c + g_0$ is $w(B, B^*)$ -lower semicontinuous. From [2, Proposition 1.3, p. 892], B is reflexive, and so $f_c + g_0$ attains a minimum at some $b \in B$. Thus $\partial(f_c + g_0)(b) \ni 0$. Since g_0 is continuous, Lemma 3.2 implies that $\partial f_c(b) + \partial g_0(b) \ni 0$, and so there exists $d^* \in \partial f_c(b)$ such that $-d^* \in \partial g_0(b)$. Since ι is surjective, there exists $d \in B$ such that $\iota(d) = d^*$. Then, from (3),

$$g_0(b) + g_0(d) = -[b, d] \quad (9)$$

and, using (5), $f_c(b) + f_c^\circ(d) = f_c(b) + f_c^*(d^*) = \langle b, d^* \rangle = [b, d]$. Combining this with (7), $f(b+c) + f^\circ(d+c) = [b+c, d+c]$. Since this gives $f(b+c) \in \mathbb{R}$, $c \in \text{dom } f - b$. From (8), $q(b+c) + q(d+c) \leq [b+c, d+c]$, or equivalently, $q(b) + q(d) \leq [b, d]$. Adding this to (9), $(g_0 + q)(b) + (g_0 + q)(d) \leq 0$. From (4), $(g_0 + q)(b) = 0$, that is to say, $b \in \mathcal{N}_q(g_0)$. Thus $c \in \text{dom } f - \mathcal{N}_q(g_0)$, as required. \square

Remark 3.4. While Theorem 3.3 is adequate for the application in Lemma 4.3, if we think of ∂f as a multifunction from B into B^* , then the proof above actually establishes that $D(\partial f) - \mathcal{N}_q(g_0) = B$.

4 Linear q -positive sets

We now come to the main topic of this paper: linear q -positive sets. Let $(B, [\cdot, \cdot], q, \|\cdot\|, \iota)$ be an SSDB space and A be a linear subspace of B . Then we write A^0 for the linear subspace $\{b \in B: [A, b] = \{0\}\}$ of B .

Lemma 4.1. *Let $B = (B, [\cdot, \cdot], q, \|\cdot\|, \iota)$ be a SSDB space and A be a maximally q -positive linear subspace of B . Then A^0 is a q -negative linear subspace.*

Proof. If $p \in A^0$ then $\inf q(A - p) = \inf q(A) + q(p) = q(p)$, and so (2) gives $q(p) \leq 0$. If $b, c \in A^0$ then $b - c \in A^0$ and so $q(b - c) \leq 0$. \square

Lemma 4.2. *Let $B = (B, [\cdot, \cdot], q, \|\cdot\|, \iota)$ be a SSDB space and A be a norm-closed q -positive linear subspace of B . Define the function $q_A: B \rightarrow]-\infty, \infty]$ by $q_A := q$ on A and $q_A := \infty$ on $B \setminus A$. Then $q_A \in \mathcal{PCLSC}(B)$ and*

$$q_A(b) + q_A^\circ(d) = [b, d] \implies b - d \in A^0. \quad (10)$$

Proof. Suppose that $a, c \in A$ and $\lambda \in]0, 1[$. Then

$$\lambda q(a) + (1 - \lambda)q(c) - q(\lambda a + (1 - \lambda)c) = \lambda(1 - \lambda)q(a - c) \geq 0.$$

It is easily seen that this implies the convexity of q_A . (See [5, Lemma 19.7, pp. 80–81].) We know from (1) that q is continuous. Since A is norm-closed in B , $q_A \in \mathcal{PCLSC}(B)$.

We now establish (10). Let $q_A(b) + q_A^\circ(d) = [b, d]$, $a \in A$ and $\lambda \in \mathbb{R}$. Then clearly $b \in A$ and $b + \lambda a \in A$, and so $[b + \lambda a, d] - [\lambda a, b] - \lambda^2 q(a) = q(b) + [b + \lambda a, d] - q(b + \lambda a) = q_A(b) + [b + \lambda a, d] - q_A(b + \lambda a) \leq q_A(b) + q_A^\circ(d) = [b, d]$. Thus $\lambda^2 q(a) + \lambda[b - d, a] \geq 0$. Since this inequality holds for all $\lambda \in \mathbb{R}$, $[b - d, a] = 0$. Since this equality holds for all $a \in A$, we obtain (10). \square

A Brezis–Browder theorem for SSDB spaces

Lemma 4.3. *Let $B = (B, [\cdot, \cdot], q, \|\cdot\|, \iota)$ be a SSDB space, A be a norm-closed q -positive linear subspace of B , and A^0 be q -negative. Then:*

- (a) $A - \mathcal{N}_q(g_0) = B$.
- (b) A is maximally q -positive.

Proof. (a) It is clear from (10) that $q_A(b) + q_A^\circ(d) = [b, d] \implies q(b - d) \leq 0$, that is to say, $q_A(b) + q_A^\circ(d) = [b, d] \implies q(b) + q(d) \leq [b, d]$. (a) now follows easily from Theorem 3.3 with $f = q_A$.

(b) Now suppose that $c \in B$ and $A \cup \{c\}$ is q -positive. From (a), there exists $a \in A$ such that $a - c \in \mathcal{N}_q(g_0)$. Thus $\frac{1}{2}\|a - c\|^2 + q(a - c) = 0$. Since $A \cup \{c\}$ is q -positive, $q(a - c) \geq 0$, and so $\frac{1}{2}\|a - c\|^2 \leq 0$, from which $c = a \in A$. This completes the proof of (b). \square

Our next result is suggested by Yao, [8, Theorem 2.4, p. 3].

Theorem 4.4. *Let $B = (B, [\cdot, \cdot], q, \|\cdot\|, \iota)$ be a SSDB space and A be a norm-closed q -positive linear subspace (of $(B, [\cdot, \cdot], q, \|\cdot\|, \iota)$). Then:*

- (a) A is maximally q -positive if, and only if, A^0 is q -negative.
- (b) A is maximally q -positive if, and only if, A^0 is maximally q -negative.

Proof. (a) is clear from Lemmas 4.1 and 4.3(b). “If” in (b) is immediate from “If” in (a). Conversely, let us suppose that A is maximally q -positive. We know already from (a) that A^0 is q -negative, and it only remains to prove the maximality. Since A is norm-closed, it follows from standard functional analysis and the surjectivity of ι that $A = (A^0)^0$. Now A^0 is $(-q)$ -positive and norm-closed. Furthermore, $(A^0)^0 = A$ is q -positive and thus $(-q)$ -negative. If we now apply Lemma 4.3(b), with A replaced by A^0 and q replaced by $-q$, we see that A^0 is maximally $(-q)$ -positive, that is to say, maximally q -negative, which completes the proof of (b). \square

Now let A be a monotone linear subspace of $E \times E^*$. Then the linear subspace A^* of $E \times E^*$ is defined by: $(x, x^*) \in A^* \iff$ for all $(a, a^*) \in A$, $\langle x, a^* \rangle = \langle a, x^* \rangle$. It is clear then that $A^0 = \rho_1(A^*)$. Corollary 4.5(a) below appears in Brezis–Browder [1, Theorem 2, pp. 32–33], and Corollary 4.5(b) in Yao, [8, Theorem 2.4, p. 3].

Corollary 4.5. *Let E be a nonzero reflexive Banach space with topological dual E^* and A be a norm-closed monotone linear subspace of $E \times E^*$. Then:*

- (a) A is maximally monotone if, and only if, A^* is monotone
- (b) A is maximally monotone if, and only if, A^* is maximally monotone.

Proof. These results follow from Theorem 4.4 and the comments in Example 2.3. \square

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A Brezis–Browder theorem for SSDB spaces

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